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SELF-DUAL HOPF QUIVERS

Hua-Lin Huang
Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, China and Mathematics Section, The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

Libin Li
Department of Mathematics, Yangzhou University, Yangzhou, Jiangsu, China

Yu Ye
Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, China

We study self-dual coradically graded pointed Hopf algebras with a help of the dual Gabriel theorem for pointed Hopf algebras (van Oystaeyen and Zhang, 2004). The co-Gabriel Quivers of such Hopf algebras are said to be self-dual. An explicit classification of self-dual Hopf quivers is obtained. We also prove that finite dimensional pointed Hopf algebras with self-dual graded versions are generated by group-like and skew-primitive elements as associative algebras. This partially justifies a conjecture of Andruskiewitsch and Schneider (2000) and may help to classify finite dimensional self-dual coradically graded pointed Hopf algebras.

Key Words: Hopf algebra; Quiver.

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1. INTRODUCTION

One can start with quivers to construct path algebras and their quotient algebras. This produces finite dimensional elementary algebras in an exhaust way, due to a well-known theorem of Gabriel. See Auslander et al. (1995) and Ringel (1984). There is a dual analog for coalgebras given by Chin and Montgomery (1997), which is remarkable for removing the restriction of finite dimensionality. Namely, any pointed coalgebra is a large subcoalgebra of the path coalgebra of some unique quiver.

A graded coalgebra $C = \bigoplus_{n\geq 0} C^n$ is said to be coradically graded, provided that $C_n = \bigoplus_{i\leq n} C^i$, where $\{C_n\}$ is the coradical filtration of C (see Chin and Musson,

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Address correspondence to Hua-Lin Huang, Mathematics Section, The Abdus Salam International Centre for Theoretical Physics, Strada Costiera 11, 34100, Trieste, Italy; Fax: +39-40-224163; E-mail: hhuang@ictp.trieste.it

1996). Throughout this article, by a graded Hopf algebra H we mean that a Hopf algebra $H = \bigoplus_{n\geq 0} H^n$ is a positively graded algebra, such that the comultiplication, the counit, and the antipode of H preserve the grading. Such an H is said to be locally finite dimensional, provided that all homogeneous spaces H^n's are finite dimensional. A locally finite dimensional graded Hopf algebra $H = \bigoplus_{n\geq 0} H^n$ is said to be self-dual, provided that there is a graded Hopf algebra isomorphism $H \cong H^{\mathrm{gr}}$, where $H^{\mathrm{gr}} = \bigoplus_{n\geq 0} H^{n*}$ is the graded dual of H. The self-duality is very natural: for any locally finite dimensional graded Hopf algebra H, the tensor product $H \otimes H^{\mathrm{gr}}$ is self-dual. We remark that self-dual Hopf algebras generated in degrees zero and 1 were studied by Green and Marcos (2000), where descriptions of such Hopf algebras via the so-called self-dual Hopf bimodules were obtained.

A quiver Q is said to be a self-dual Hopf quiver, provided that Q is the (co-Gabriel) quiver $Q(H)$ (see 2.4 below) of a self-dual, coradically graded, pointed Hopf algebra H. It can be characterized as the Hopf quiver of a finite abelian group with a finite ramification, or equivalently, the quiver of a (finite dimensional) self-dual Hopf bimodule of a finite abelian group (see 2.2 below). We also prove that a finite dimensional pointed Hopf algebra H, with $\mathrm{gr}\, H$, the graded version of H (see 2.4 below), being self-dual, is generated by group-like and skew-primitive elements. This partially justifies a conjecture of Andruskiewitsch and Schneider (2000, Conjecture 1.4).

For simplicity of exposition, we assume throughout that k is an algebraically closed field of characteristic zero. All algebras and coalgebras are over k. For a finite dimensional vector space V, we denote its k-linear dual by V^*. Unendowed tensor product $\otimes$ is $\otimes_k$.

2. HOPF QUIVERS AND DUAL GABRIEL'S THEOREM

We begin by recalling general facts, due to Cibils and Rosso (2002) and van Oystaeyen and Zhang (2004), about constructing graded Hopf structures from path coalgebras and the dual Gabriel's theorem for pointed Hopf algebras.

2.1.

Let Q be a quiver and kQ the k-space with all the paths of Q as a basis. Then kQ has a natural length gradation $kQ = \bigoplus_{n\geq 0} kQ_n$, where kQ_n is spanned by all the paths of length n. Note that Q_0 is the set of vertices and Q_1 is the set of arrows. For each nontrivial path $p = a_n \cdots a_2 a_1 \in Q_n$ (i.e., $n \geq 1$) we define its starting vertex $s(p)$ as the tail of arrow a_1 and terminating vertex $t(p)$ as the head of arrow a_n.

Given a quiver Q, the graded space kQ has a natural graded path coalgebra structure as follows:

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1 \quad \text{for each } g \in Q_0,$$
$$\Delta(p) = t(p) \otimes p + a_n \otimes a_{n-1} \cdots a_1 + \cdots + a_n \cdots a_2 \otimes a_1 + p \otimes s(p), \qquad \varepsilon(p) = 0$$
$$\text{for each nontrivial path } p = a_n \cdots a_1.$$

It is obvious that kQ is pointed with set of group-like elements $G(kQ) = Q_0$, and has the following coradical filtration

$$kQ_0 \subseteq kQ_0 \oplus kQ_1 \subseteq kQ_0 \oplus kQ_1 \oplus kQ_2 \subseteq \cdots .$$

Hence kQ is coradically graded. We remark that the path coalgebra kQ has another presentation as the so-called cotensor coalgebra and hence enjoys a universal property (see Nichols, 1978; van Oystaeyen and Zhang, 2004).

2.2.

Let G be a group and $\mathscr{C}$ the set of its conjugacy classes. A ramification datum R of the group G is a class function $\sum_{C\in\mathscr{C}} R_C C$ with coefficients in $\mathbb{N}_0 \cup \{\infty\}$, where $\mathbb{N}_0$ is the set of non-negative integers. A ramification R of G is said to be finite, provided that all the R_C's are finite. Recall that for each ramification datum R of G, the corresponding Hopf quiver $Q = Q(G, R)$ is defined as follows: The set of vertices Q_0 is G, and for each $x \in G$ and $c \in C$, there are R_C arrows from x to cx.

A vector space M is said to be a kG-Hopf bimodule if it is simultaneously a kG-bimodule and a kG-bicomodule such that the comodule structure maps are homomorphisms of kG-bimodules.

Hopf bimodules over kG were classified in Cibils and Rosso (1997, Proposition 3.3). We briefly recall this result for later application. For each $C \in \mathscr{C}$, fix an element $u(C) \in C$, and let Z_C be the centralizer of $u(C)$. There is an equivalence of categories

$$V : \mathrm{b}(kG) \longrightarrow \prod_{C\in\mathscr{C}} \mathrm{mod}(kZ_C),$$

where $\mathrm{b}(kG)$ is the category of kG-Hopf bimodules and $\mathrm{mod}(kZ_C)$ the category of left kZ_C-modules. Given $M \in \mathrm{b}(kG)$, then $V(M) = ({}^{u(C)}M^1)_{C\in\mathscr{C}}$, where the left module structure on ${}^{u(C)}M^1$ is defined by the conjugate action: $g \cdot m = g \cdot m \cdot g^{-1}$. On the contrary, for any $(M_C)_{C\in\mathscr{C}} \in \Pi_{C\in\mathscr{C}} \mathrm{mod}(kZ_C)$, the corresponding kG-Hopf bimodule is $\bigoplus_{C\in\mathscr{C}} kG \bigotimes_{kZ_C} M_C \otimes kG$.

Given a kG-Hopf bimodule M with bicomodule maps δ_L and δ_R, we define the quiver $Q = Q(G, M)$ of M as follows: The set of vertices Q_0 is G, and for any $g, h \in G$, there are $\dim_k {}^hM^g$ arrows from g to h. Here by ${}^hM^g$ we mean the (h, g)-isotypic component

$$\{m \in M \parallel \delta_L(m) = h \otimes m, \delta_R(m) = m \otimes g\}.$$

The following lemma shows that the Hopf quivers arising from ramification data coincide with those quivers from Hopf bimodules over a group, hence we may identify them by just saying Hopf quivers.

Lemma 2.1. *For any quiver $Q = Q(G, M)$, there exists a ramification datum R of G such that $Q = Q(G, R)$, and vice versa.*

Proof. Let M be a kG-Hopf bimodule with comodule structure maps δ_L and δ_R and $Q = Q(G, M)$. For any $f, g, h \in G$ and $m \in {}^hM^g$, by the definition of kG-Hopf bimodules, we have

$$\delta_L(f \cdot m) = fh \otimes f \cdot m, \qquad \delta_L(m \cdot f) = hf \otimes m \cdot f$$

and

$$\delta_R(f \cdot m) = f \cdot m \otimes fg, \qquad \delta_R(m \cdot f) = m \cdot f \otimes gf.$$

It follows that

$$f \cdot {}^hM^g \subseteq {}^{fh}M^{fg}, \qquad {}^hM^g \cdot f \subseteq {}^{hf}M^{gf}.$$

Note that f is invertible, hence actually we have

$$f \cdot {}^hM^g = {}^{fh}M^{fg}, \qquad {}^hM^g \cdot f = {}^{hf}M^{gf}.$$

It follows that for $x, g, c \in G$,

$${}^{g^{-1}cgx}M^x = {}^{g^{-1}cg}M^1 \cdot x = g^{-1} \cdot {}^cM^1 \cdot g \cdot x.$$

Since the actions of group elements are invertible, it is clear that

$$\dim_k {}^{g^{-1}cgx}M^x = \dim_k {}^cM^1.$$

In other words, for any $x \in G$ and any $c' \in C$, where C is the conjugacy class containing c, there are $\dim_k {}^cM^1$ arrows from x to $c'x$ in Q. Let $\mathscr{C}$ be the set of the conjugacy classes of G. For each $C \in \mathscr{C}$, fix an element $c \in C$. Take a ramification data of G as

$$R = \sum_{C \in \mathscr{C}} R_C C$$

with $R_C = \dim_k {}^cM^1$. It is clear that $Q = Q(G, R)$.

On the contrary, let $Q = Q(G, R)$ for some $R = \sum_{C \in \mathscr{C}} R_C C$. Take $(M_C)_{C \in \mathscr{C}} \in \Pi_{C \in \mathscr{C}} \mathrm{mod}(kZ_C)$ such that $\dim_k M_C = R_C$. This is always possible. For example, take M_C as trivial kZ_C-module. Let M be the associated kG-Hopf bimodule. By direct calculation of the isotypic components of M, we have that $Q = Q(G, M)$. $\square$

2.3.

Suppose that kQ can be endowed with a graded Hopf algebra structure with length gradation. Then kQ is pointed and kQ_0 is the coradical. Hence $kQ_0 \cong kG$ for some finite group G and we now identify Q_0 and G. The graded Hopf algebra structure induces naturally on kQ_1 a kG-Hopf bimodule structure and Q is of course the Hopf quiver of it. By Lemma 2.1, the quiver Q is the Hopf quiver $Q(G, R)$ of some ramification data R.

Given a Hopf quiver $Q = Q(G, R)$ for some group G and some ramification data R, then kQ_1 admits kQ_0-Hopf bimodule structures. Fix a kQ_0-Hopf bimodule $(kQ_1, m_L, m_R, \delta_L, \delta_R)$. By the universal property of kQ, the bimodule structure can be extended to an associative multiplication and kQ becomes a graded bialgebra. The existence of antipode is guaranteed by a result of Takeuchi (1971). Hence kQ admits a graded Hopf structure.

Cibils and Rosso's results (2002) can be summarized as follows.

Theorem 2.2. *Let Q be a quiver. Then Q is a Hopf quiver if and only if the path coalgebra kQ admits graded Hopf algebra structures, if and only if Q_0 is a group and kQ_1 admits a kQ_0-Hopf bimodule structure.*

2.4.

Let C be a pointed coalgebra with $G = G(C)$, then the corresponding quiver $Q(C)$ is obtained in the following way. The set of vertices of $Q(C)$ is G. For $\forall\, x, y \in G$, the number of arrows from x to y is $\dim_k P_{x,y}(C) - 1$, where $P_{x,y}(C) = \Delta^{-1}(C \otimes x + y \otimes C)$. Chin and Montgomery's (1997) theorem says that C is a large subcoalgebra of the path coalgebra $kQ(C)$. Here "large" means that the subcoalgebra contains all the vertices and arrows of $Q(C)$. Of course, in this case such a quiver is unique, called the co-Gabriel quiver of C. We remark that, according to the definition, a pointed coalgebra and its associated graded coalgebra (induced by the coradical filtration) enjoy the same quiver.

Let H be a pointed Hopf algebra. The coradical filtration $\{H_n \mid n \geq 0\}$ is in fact a Hopf algebra filtration and hence the associated graded space

$$\operatorname{gr} H = \bigoplus_{n \geq 0} (\operatorname{gr} H)^n = \bigoplus_{n \geq 0} H_n / H_{n-1}$$

(with $H_{-1} = 0$) is a coradically graded Hopf algebra (see Montgomery, 1993, Lemma 5.2.8). Consider the quiver $Q(H)$, called the co-Gabriel quiver, of the underlying coalgebra of H. The following result can be regarded as the version of the Gabriel's theorem for Hopf algebras from the coalgebra aspect, see van Oystaeyen and Zhang (2004, Proposition 4.4 and Theorem 4.6).

Theorem 2.3. *Suppose that H is a pointed Hopf algebra and that $G = G(H)$. Then $Q(H)$ is a Hopf quiver and there exists a graded Hopf algebra embedding $\operatorname{gr} H \hookrightarrow kQ(H)$, where the Hopf structure on $kQ(H)$ is determined by the kG-Hopf bimodule structure on $(\operatorname{gr} H)^1$.*

3. SELF-DUAL HOPF QUIVERS

In this section, we consider the co-Gabriel quivers of self-dual coradically graded pointed Hopf algebras, which are called self-dual Hopf quivers.

3.1.

Let $H = \bigoplus_{n \geq 0} H^n$ be a positively graded Hopf algebra. In this section, we always assume that H is locally finite dimensional, namely, the homogeneous spaces are finite dimensional. The Hopf algebra H is said to be self-dual if there exists a graded Hopf isomorphism $H \cong H^{\mathrm{gr}}$, where $H^{\mathrm{gr}} = \bigoplus_{n \geq 0} H^{n*}$ is the graded dual of H.

A Hopf quiver Q is said to be self-dual if $Q = Q(H)$ for some self-dual coradically graded pointed Hopf algebra H.

For an explicit description of the self-dual Hopf quivers, we need the notion of self-dual Hopf bimodules over a finite Abelian group algebra. Let G be a finite Abelian group. A finite dimensional vector space M is said to be a self-dual kG-Hopf bimodule if it is a kG-Hopf bimodule and there is an isomorphism of kG-Hopf bimodules $M \cong M^*$, where M^*'s kG-Hopf bimodule structure is induced by a Hopf isomorphism $kG \cong (kG)^*$.

The main result of this section is the following theorem.

Theorem 3.1. *The following are equivalent*:

(1) *Q is a self-dual Hopf quiver*;
(2) *Q is the Hopf quiver of a finite Abelian group with a finite ramification*;
(3) *$Q = Q(G, M)$, where G is a finite Abelian group, and M is a (finite dimensional) self-dual kG-Hopf bimodule.*

3.2.

The classification of finite dimensional self-dual Hopf bimodules over a finite Abelian group algebra was given in Green and Marcos (2000) using Cibils and Rosso's results (1997). We recall it here for application later on.

Let G be a finite Abelian group. Write $G = G_1 \times G_2 \times \cdots \times G_t$, where $G_i = \langle \alpha_i \rangle$. The general elements of G are written as $\alpha^e = \alpha_1^{e_1} \cdot \alpha_2^{e_2} \cdots \alpha_t^{e_t}$. Let $\omega = \{\omega_1, \omega_2, \ldots, \omega_t\}$ be a set of roots of unity such that order ω_i = order α_i. We define a map $\chi^\omega : kG \longrightarrow (kG)^*$ as follows: for any element $\alpha^e \in G$, let $\chi^\omega(\alpha^e) = \chi^\omega_{\alpha^e} \in (kG)^*$; for any $\alpha^f \in G$, let $\chi^\omega_{\alpha^e}(\alpha^f) = \omega_1^{e_1 f_1} \omega_2^{e_2 f_2} \cdots \omega_t^{e_t f_t}$. It is well-known that such a map χ^ω is a Hopf isomorphism and that $\{\chi^\omega_g\}_{g \in G}$ is a complete set of irreducible characters of G. Denote by S_g the irreducible module associated to the character χ^ω_g.

By Cibils and Rosso's (1997) classification of Hopf bimodules, there is an equivalence of categories

$$V : \mathrm{b}(kG) \longrightarrow \prod_{g \in G} \mathrm{mod}(kG),$$

where $\mathrm{b}(kG)$ is the category of kG-Hopf bimodules and $\mathrm{mod}(kG)$ the category of left kG-modules. Given $M \in \mathrm{b}(kG)$, then $V(M) = ({}^g M^1)_{g \in G}$. Write ${}^g M^1 = \bigoplus_{h \in G} m_h(g) S_h$ as the sum of irreducible modules. Then the isomorphic classes of objects in $\mathrm{b}(kG)$ are in one-to-one correspondence with the set of matrices

$$\{(m_h(g))_{g,h \in G} \mid m_h(g) \text{ is a nonnegative integer, } \forall g, h \in G\}.$$

Identifying kG with $(kG)^*$ via χ^ω, then M^* is a kG-Hopf bimodule. By Cibils and Rosso (1997, Proposition 5.1), if M corresponds to the matrix $(m_h(g))_{g,h \in G}$, then M^* corresponds to the matrix $(m^*_h(g))_{g,h \in G}$, where $m^*_h(g) = m_{g^{-1}}(h^{-1})$.

Now it is clear that a kG-Hopf bimodule M is self-dual if and only if there exists an ω as in the previous argument such that the corresponding matrix $(m_h(g))_{g,h \in G}$ of M satisfying $m_h(g) = m_{g^{-1}}(h^{-1})$, for any $g,\ h \in G$.

3.3.

Proof of Theorem 3.1. (1) $\Rightarrow$ (2) Assume that $Q = Q(H)$, where $H = \bigoplus_{n \geq 0} H^n$ is a self-dual coradically graded pointed Hopf algebras. By the assumption, $H^0 = kG$

for some finite group and H^1 is a kG-Hopf bimodule. In fact, $Q(H) = Q(G, H^1)$. By Lemma 2.1, it is enough to show that G is Abelian. Since H is self-dual, hence there exists a grade Hopf isomorphism $f : H \longrightarrow H^{\mathrm{gr}}$. Consider the restriction f_0 of f to degree zero. Clearly, f_0 gives a Hopf isomorphism between kG and $(kG)^*$. It follows that G is Abelian since $(kG)^*$ is commutative.

(2) $\Rightarrow$ (3) Assume that $Q = Q(G, R)$, where G is a finite Abelian group and $R = \sum_{g\in G} R_g g$ a finite ramification datum. Then by Lemma 2.1, $Q = Q(G, M)$ for any Hopf bimodule M such that $\dim_k {}^gM^1 = R_g$, $\forall g \in G$. It suffices to prove that there exists a self-dual Hopf bimodule satisfying such condition. For this, we fix an ω as in Subsection 3.2. Let M be the kG-Hopf bimodule corresponding to matrix $(m_h(g))_{g,h\in G}$ with entries $m_{g^{-1}}(g) = R_g$, $\forall g \in G$ and zero otherwise. It is clear that such an M is self-dual, by the argument of Subsection 3.2.

(3) $\Rightarrow$ (1) Assume that $Q = Q(G, M)$, where G is a finite Abelian group, and M is a self-dual kG-Hopf bimodule. Now kQ has a coradically graded Hopf structure determined by the self-dual kG-Hopf bimodule M. Let H be the Hopf subalgebra of kQ generated by Q_0 and Q_1. By Montgomery (1993, Theorem 2.2), H is self-dual. It is clear that H is coradically graded pointed and $H^0 = kG$ and $H^1 = M$. Hence by Subsection 2.4, $Q = Q(H)$. That is, Q is the co-Gabriel quiver of the self-dual coradically graded pointed Hopf algebra H, and hence a self-dual Hopf quiver. □

4. POINTED HOPF ALGEBRA *H* WITH gr *H* BEING FINITE DIMENSIONAL SELF-DUAL

The main purpose of this section is to prove that a finite dimensional pointed Hopf algebra H is generated by its group-like and skew-primitive elements if its graded version $\mathrm{gr}\, H$ is self-dual.

4.1.

Andruskiewitsch and Schneider (1998) proposed the so-called lifting method for classifying finite dimensional pointed Hopf algebras. In the program, a key step is to find the (nice) generators. Andruskiewitsch and Schneider conjectured that all finite dimensional pointed Hopf algebras over an algebraically closed field of characteristic zero are generated by group-like and skew-primitive elements (see Andruskiewitsch and Schneider, 2000, Conjecture 1.4).

The following theorem shows that Andruskiewitsch and Schneider's conjecture is true for finite dimensional pointed Hopf algebras with self-dual graded versions.

Theorem 4.1. *Let H be a finite dimensional pointed Hopf algebra. If* $\mathrm{gr}\, H$ *is self-dual, then H is generated by group-like and skew-primitive elements.*

Proof. Let $\{H_n\}$ be the coradical filtration of H. Its graded version

$$\mathrm{gr}\, H = \bigoplus_{n\geq 0} (\mathrm{gr}\, H)^n = \bigoplus_{n\geq 0} H_n/H_{n-1}$$

(with $H_{-1} = 0$) is a coradically graded Hopf algebra. By the assumption of $\operatorname{gr} H$ being self-dual, it follows by Theorem 3.1 that $(\operatorname{gr} H)^0 = kG$ for some finite Abelian group G. Let $J = \bigoplus_{n\geq 1}(\operatorname{gr} H)^n$. It is clear that J is a nilpotent (Hopf) ideal of $\operatorname{gr} H$. Note that $\operatorname{gr} H/J = (\operatorname{gr} H)^0 = kG$, which is isomorphic to $k^{|G|}$ as an associative algebra. It follows that $\operatorname{gr} H$ is an elementary algebra and J is the Jacobson radical. It is clear that $J^2 \subseteq \bigoplus_{n\geq 2}(\operatorname{gr} H)^n$, and hence $(\operatorname{gr} H)^1 \subseteq J/J^2$.

On the other hand, by the duality of coradical filtration and Jacobson radical series (see e.g., Montgomery, 1993, 5.2.9), we have $J^2 = C_1((\operatorname{gr} H)^*)^{\perp}$, where $C_1((\operatorname{gr} H)^*)$ is the first term of the coradical filtration of the dual Hopf algebra $(\operatorname{gr} H)^*$. By the self-duality of $\operatorname{gr} H$, $C_1((\operatorname{gr} H)^*) = C_1(\operatorname{gr} H) = (\operatorname{gr} H)^0 \oplus (\operatorname{gr} H)^1$ since $\operatorname{gr} H$ is coradically graded. This implies that $\dim_k J^2 = \dim_k \operatorname{gr} H - \dim_k(\operatorname{gr} H)^0 - \dim_k(\operatorname{gr} H)^1$.

By comparing the dimensions, we have $J^2 = \bigoplus_{n\geq 2}(\operatorname{gr} H)^n$, and hence $(\operatorname{gr} H)^1 = J/J^2$. It is well known that (see e.g., Auslander et al., 1995, Theorem 1.9, p. 65), as an associative algebra, $\operatorname{gr} H$ is generated by $\operatorname{gr} H/J$ and J/J^2. This deduces that H is generated by H_1, by applying Lemma 2.2 of Andruskiewitsch and Schneider (1998). Now the theorem follows by Taft-Wilson Theorem (see e.g., Montgomery, 1993, Theorem 5.4.1, p. 68). □

Remark 4.2. Let $H = \bigoplus_{n\geq 0} H^n$ be a finite dimensional self-dual coradically graded pointed Hopf algebra. By J we denote its Jacobson radical. Then by a similar argument of comparing dimensions, via the duality of coradical filtration and Jacobson radical series, we have $J^m = \bigoplus_{n\geq m} H^n$, for any integer $m \geq 1$.

4.2.

Let $Q = Q(G, M)$ be a Hopf quiver and kQ the graded Hopf algebra determined by the kG-Hopf bimodule M. By $kG[M]$ we denote the Hopf subalgebra of kQ generated by Q_0 and Q_1. This is actually the so-called bialgebra of type one, in the sense of Nichols (1978).

An immediate consequence of Theorem 4.1 says that any finite dimensional self-dual coradically graded pointed Hopf algebra is a bialgebra of type one.

Corollary 4.3. *Any finite dimensional self-dual coradically graded pointed Hopf algebra is of the form $kG[M]$ for some finite Abelian group G and some self-dual kG-Hopf bimodule M.*

4.3.

Finally, we remark that there is not known necessary and sufficient condition for general self-dual kG-Hopf bimodule M such that $kG[M]$ is finite dimensional. However we work out the simplest case with a help of results in Chen et al. (2004).

Let G be a cyclic group of order n generated by g. Let $R = g$ be the simplest ramification datum. Then the Hopf quiver $Q = Q(G, R)$ is a basic cycle. Namely, Q has set of vertices $\{g^i | i = 0, 1, \ldots, n-1\}$ and set of arrows $\{a_i : g^i \longrightarrow g^{i+1} \mid i = 0, 1, \ldots, n-1\}$. Finite dimensional pointed Hopf algebras with co-Gabriel quiver Q are completely classified in Chen et al. (2004, Theorem 3.6). As a consequence we have

Proposition 4.4. *Let H be a finite dimensional pointed Hopf algebra with $Q(H)$ being a basic cycle. Then H is self-dual if and only if H is the Taft algebra.*

Proof. Recall that the Taft algebra T of dimension n^2 is generated by two elements h and x with relations

$$x^n = 0, \qquad h^n = 1, \qquad xh = qxh,$$

where q is an nth primitive root of unity. The self-duality of T was given in Cibils (1993).

Now let H be a finite dimensional pointed Hopf algebra such that its quiver $Q(H)$ is the basic cycle Q. Then by a dual version of Lemma 2.1 in Huang et al. (2004), the Hopf algebra H is monomial (see Chen et al., 2004, Definition 1.2). Since finite dimensional Hopf algebras are co-Frobenius, hence by Lemma 2.3 in Chen et al. (2004), $H \cong C_d(n)$ as coalgebra. Now by Theorem 3.6 in Chen et al. (2004), H is isomorphic to a Hopf algebra of form $A(n, d, \mu, q)$, which is presented by generators and relations as follows

$$h^n = 1, \qquad x^d = \mu(1 - h^d), \qquad xg = ugx,$$

with u a root of unity of order d and $\mu = 0$ or 1. By Chen et al. (2004), Theorem 4.3, if $A(n, d, \mu, q)$ is self-dual, then $\mu = 0$, and $d = n$. That is, $A(n, d, \mu, q)$ must be exactly the Taft algebra. □

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